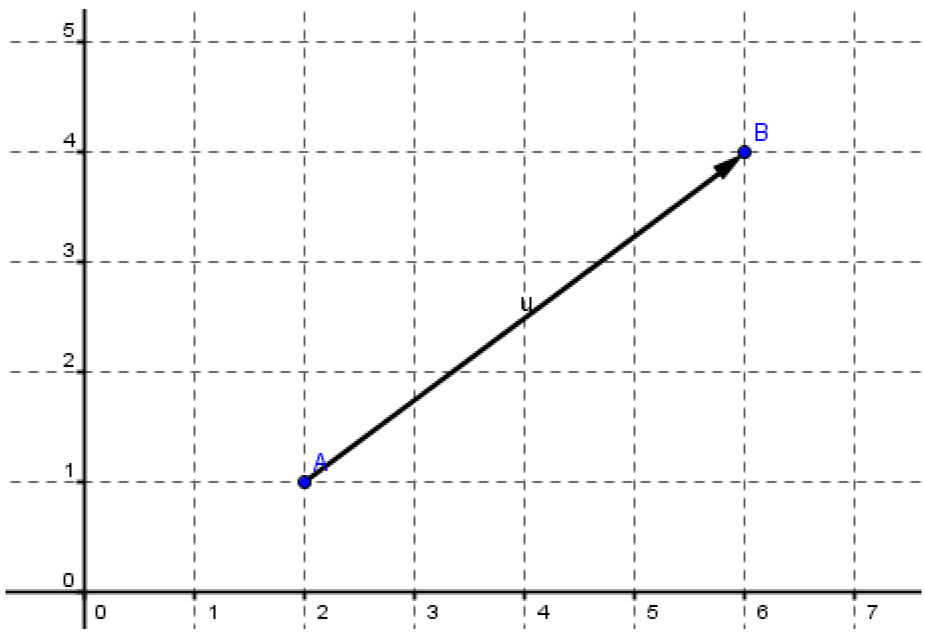
**Math 230**
**Spring 09**

## Vector Basics, with Exercises

This sheet is designed to follow the GeoGebra Introduction to Vectors. It includes a summary of some of the properties of vectors, as well as homework exercises.

### Vector Basics:

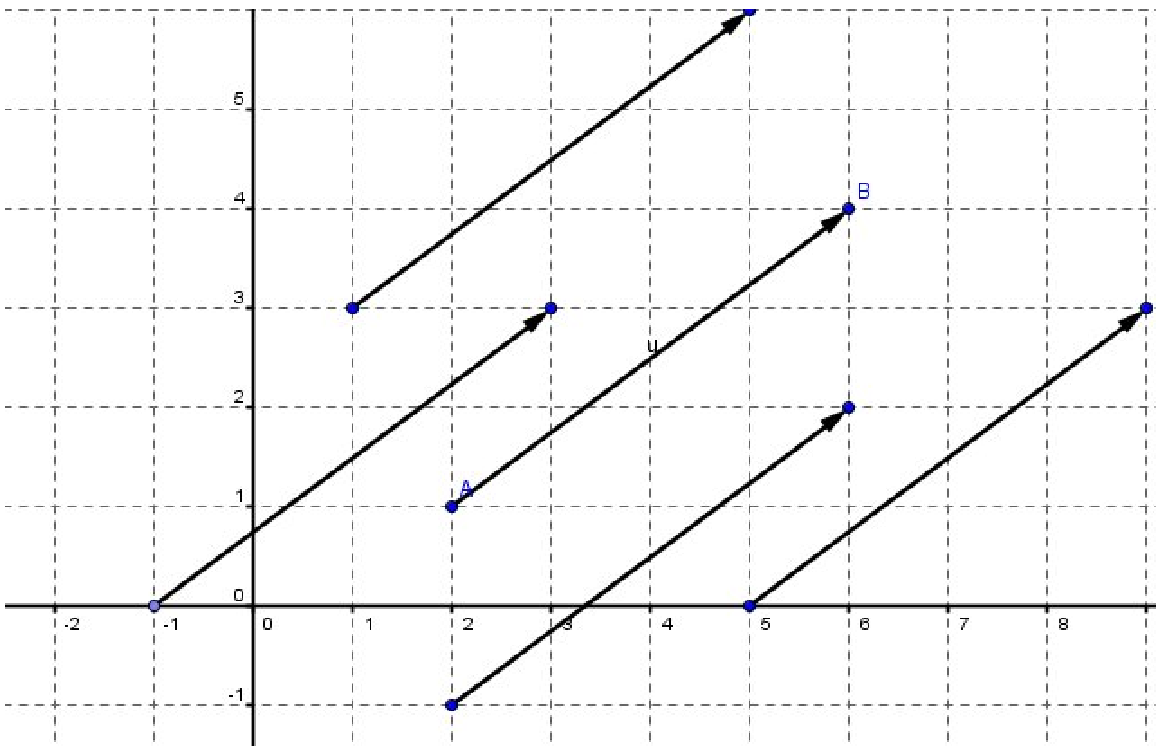
A *vector* is a displacement, which has both magnitude ("how much") and direction ("which way"). Vectors are often represented as arrows.




The vector $\vec{u} = \overrightarrow{AB}$ connects point A to point B. Note that if we translate ("slide") from point A to point B, point A would move four spaces to the right and three spaces up (negative values correspond to moving left or down, respectively). We can also represent $\vec{u} = \overrightarrow{AB}$ by its *components*; in our example, the *horizontal component of* $\vec{u}$ is 4 and the *vertical component* of $\vec{u}$ is 3, and $\vec{u} = (4,3)$. There are also other ways to describe vector displacements that we will explore later.

We can find $\overrightarrow{AB}$ from the coordinates of A and B by subtraction. If $A = (a_1, a_2)$ and $B = (b_1, b_2)$, then $\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2)$. In the example above, $A = (2,1)$ and $B = (6,4)$, so $\overrightarrow{AB} = (6-2, 4-1) = (4,3)$, as above.

Note that vectors that are parallel and the same length are all the same; all of the vectors below are equal to vector $\vec{u}$ above:



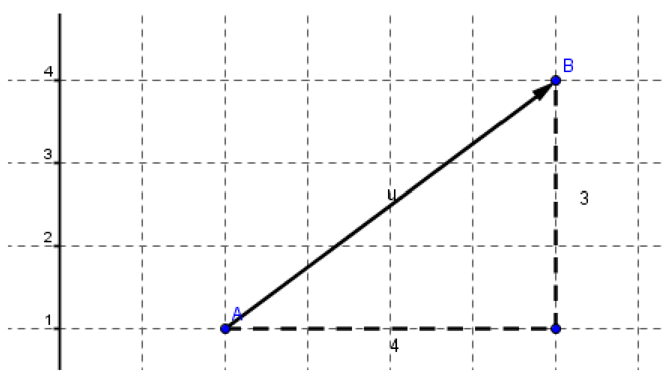

**Exercises:**

You'll need graph paper for this homework assignment.

1. Draw the vectors (-2,3) and (4,-1) in several different locations.

2. Let $A=(0,-3), B=(2,5), C=(-4,-3),$ and $D=(-2,3)$ be points.
   a. Graph the points.
   b. Find the following vectors arithmetically and graphically:
   $\overrightarrow{AB}, \overrightarrow{BA}, \overrightarrow{AD}, \overrightarrow{CD}, \overrightarrow{DC},$ and $\overrightarrow{BC}$

The length of vector $\vec{u}$ (also called the *norm*) is written $\|u\|$. To find it, we use the



Pythagorean Theorem. In the example above, we have $\|\vec{u}\|^2 = 3^2 + 4^2$ =25, so $\|\vec{u}\|^2 = 25$, and $\|\vec{u}\| = \sqrt{25} = 5$. The norm of a vector is always greater than or equal to zero (a vector can't have negative length).

**Exercises:**

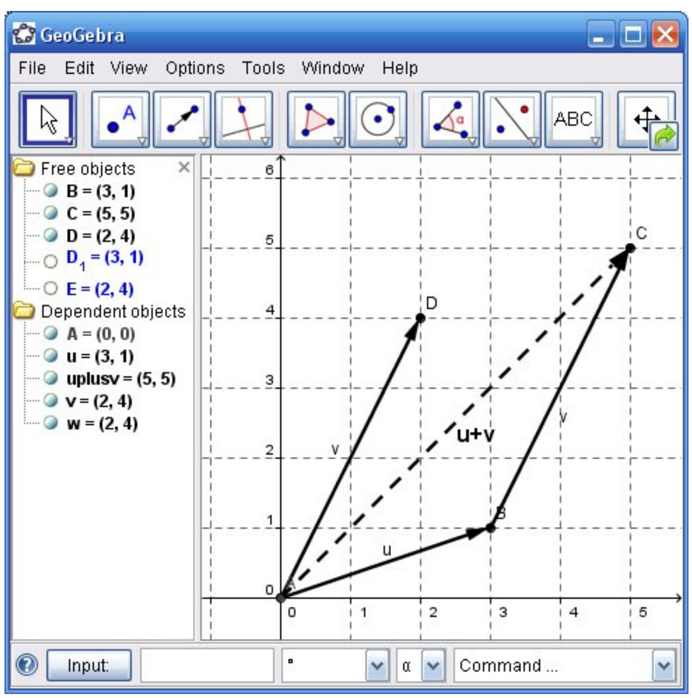
3. Find the norm of each vector in question 2.

4. For any two points A, B, will $\|\overrightarrow{AB}\| = \|\overrightarrow{BA}\|$ always be true? Explain.

**Addition and Subtraction:**

To find $\vec{u} + \vec{v}$ geometrically, we think of the vectors as displacements. First choose a starting point, A, so we have $\vec{u} = \overrightarrow{AB}$ (point A can be anywhere, but once we've chosen A, then B is fixed). Often, we will choose A to be at the origin, as we do below. Vector $\vec{u}$ takes us from A to B. To add vector $\vec{v}$, we choose point B as our starting point, and then we end up at point C, where $\vec{v} = \overrightarrow{BC}$. The sum of the two vectors, $\vec{u} + \vec{v}$, is $\overrightarrow{AC}$, the vector whose displacement is equal to the two sequential displacements.

Algebraically, we simply add the horizontal and vertical displacements: if $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$, then $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2)$. In the example below, we have $\vec{u} = (3,1)$, $\vec{v} = (2,4)$, and $\vec{u} + \vec{v} = (3+2, 1+4) = (5,5)$.




Note that two copies of $\vec{u}$ (only one is shown above) and the two copies of $\vec{v}$ form a parallelogram, and $\vec{u} + \vec{v}$ is one diagonal of this parallelogram.

**Exercises:**

Let $\vec{u} = (2,4)$, $\vec{v} = (-3,-2)$, $\vec{w} = (4,1)$, and $\vec{z} = (0,-3)$.

5. Find both geometrically and algebraically:

   a. $\vec{u} + \vec{v}$   b. $\vec{w} + \vec{z}$   c. $\vec{v} + \vec{u}$   d. $\vec{v} + \vec{z}$

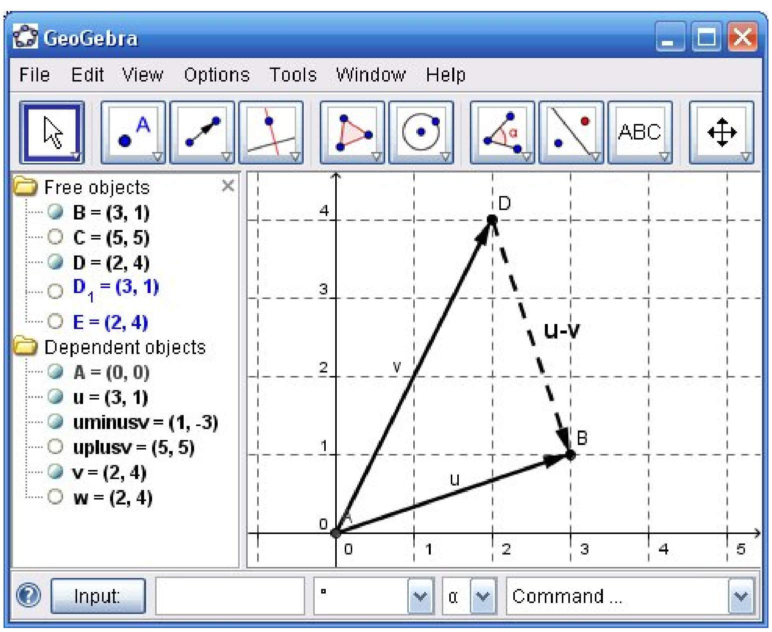
6. Find both geometrically and algebraically (use answers from the previous problem):

   a. $(\vec{u} + \vec{v}) + \vec{z}$   b. $\vec{u} + (\vec{v} + \vec{z})$

7. Did you get the same results for both of the above? Will you always get the same results for similar problems (adding three vectors in order with the parentheses switched)? Explain. Is it possible to speak without ambiguity about the sum of three vectors?

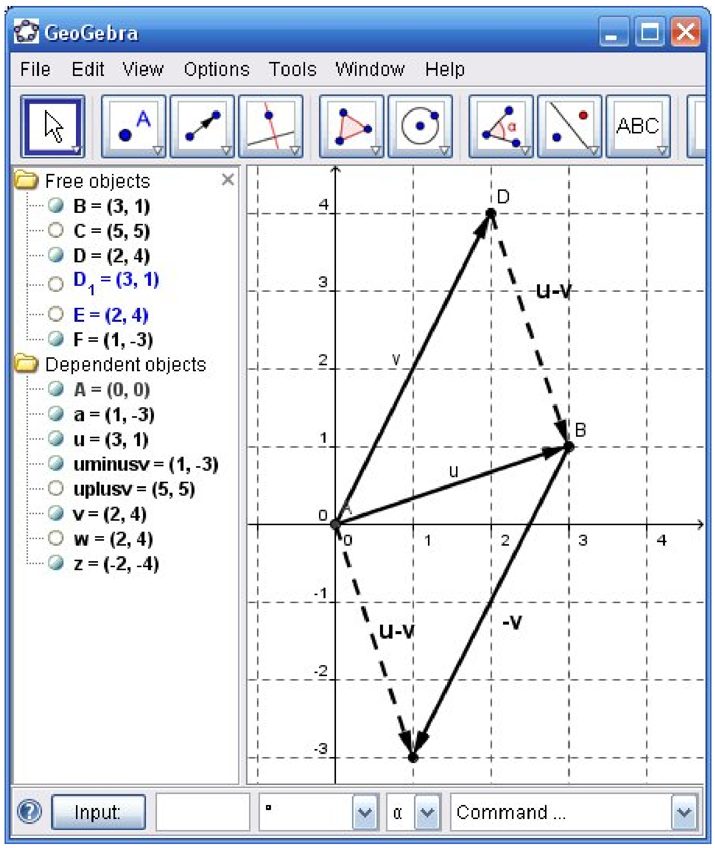
8. Did you get the same results for problem 5 parts a and c? Will you always get the same results when you switch the order of vector addends? Explain.

To think about vector subtraction, let's first think about ordinary subtraction. Suppose we want to find $10-3$. One way to think of the problem, the *missing addend* model, is to ask what we have to add to 3 to get 10, i.e. to fill in the box: $3 + \square = 10$. To find $\vec{u} - \vec{v}$, we can fill in the box with the analogous problem $\vec{v} + \square = \vec{u}$.



In the picture above, we see that $\vec{v}+(\vec{u}-\vec{v})=\vec{u}$, or that $\vec{u}-\vec{v}$, "fills in the box." Thus, one way to visualize subtracting vectors is make an arrow from the tip of the subtrahend to the tip of the minuend.

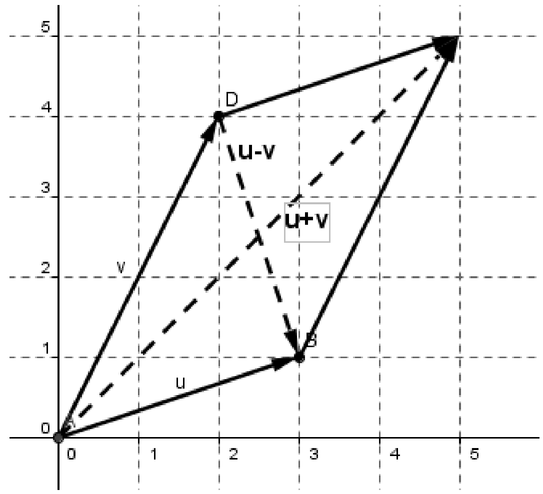
Another way to think of $10-3$ is as $10+(-3)$. We can also think of $\vec{u}-\vec{v}$ as $\vec{u}+(-\vec{v})$. What does $-\vec{v}$ mean geometrically? In arithmetic, -3 is the number that we add to 3 in order to get 0. With vectors, the analogy to 0 is a vector that doesn't displace at all, i.e. $\vec{0}=(0,0)$, and $-\vec{v}$ must "undo" any displacement "done" by $\vec{v}$. Thus $-\vec{v}$ is a vector of the same magnitude as $\vec{v}$, but pointing in the opposite direction (we just reverse the direction of the arrow). Here is $\vec{u}+(-\vec{v})$ with our example from above.




Note that the above diagram shows the results from both ways of looking at vector subtraction. In each case $\vec{u}-\vec{v}$ is the same vector, but the different viewpoints put the vector in different locations.

As we noted above, two copies of $\vec{u}$ and two copies of $\vec{v}$ form a parallelogram, and $\vec{u}+\vec{v}$ is one diagonal of the parallelogram. The other diagonal is $\vec{u}-\vec{v}$ (see below).

Once again, algebraically, computing $\vec{u}-\vec{v}$ is straightforward: if $\vec{u}=(u_1,u_2)$ and $\vec{v}=(v_1,v_2)$, then $\vec{u}-\vec{v}=(u_1-v_1,u_2-v_2)$. In the example below, we have $\vec{u}=(3,1)$, $\vec{v}=(2,4)$, and $\vec{u}-\vec{v}=(3-2,1-4)=(1,-3)$. Study the pictures and make sure you can connect the two representations in your mind.




**Exercises:**

Let $\vec{u} = (2,4), \vec{v} = (-3,-2), \vec{w} = (4,1),$ and $\vec{z} = (0,-3)$.

9. Find both geometrically and algebraically:

a. $\vec{u} - \vec{v}$   b. $\vec{w} - \vec{z}$   c. $\vec{v} - \vec{u}$   d. $\vec{v} - \vec{z}$

10. Find both geometrically and algebraically (use answers from the previous problem):

a. $(\vec{u} - \vec{v}) - \vec{z}$   b. $\vec{u} - (\vec{v} - \vec{z})$

11. Did you get the same results for both of the above? Will you always get the same results for similar problems (subtracting three vectors in order with the parentheses switched)? Explain. Is it possible to speak without ambiguity about the difference of three vectors?

12. Did you get the same results for problem 5 parts a and c? Will you always get the same results when you switch the order of subtraction? If not, will the results always be related? Explain.

**Scalar Multiplication:**

When we found $-\vec{v}$, we were finding -1 multiplied by $\vec{v}$. We call the multiplier -1 a *scalar*. Of course, -1 is also an integer, a real number, etc. In the context of vectors, we use the word *scalar* to distinguish -1 from the vectors we are working with; $\vec{v}$ is a vector, and -1 is not a vector; it's a scalar. Usually the terminology is most important when we are using variables.

Let $k$ be a scalar, that is, let $k$ stand for any real number. We can find $k\vec{u}$ for any scalar values of $k$ and vector values of $\vec{u}$. Algebraically, as has often been the case, scalar multiplication is straightforward. If $u = (u_1, u_2)$, then $ku = (ku_1, ku_2)$. Thus, if $\vec{u} = (4,-2)$, then $3\vec{u} = (3\cdot 4, 3\cdot(-2)) = (12,-6)$ and $-0.5\vec{u} = (-2,1)$.

Note that we don't right now know how to multiply two vectors, only a vector and a scalar. There is a way to multiply vectors so the result is a scalar. There is also another way to multiply vectors so that the result is a vector. In three or more dimensions, there is more than one way to multiply vectors to get another vector. Vector multiplication is more complicated than addition, subtraction, and scalar multiplication of a vector.

**Exercises:**

13. Let $\vec{u} = (4,-2)$. Find $2\vec{u}, 3\vec{u}, \frac{1}{4}\vec{u}, -\vec{u},$ and -$2\vec{u}$ algebraically.

14. Graph each of your results from the last problem. Formulate a geometric description of scalar multiplication.